Noisy Satellite Pursuit-Evasion Guidance

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Guidance laws are developed for both pursuer and evader satellites, using a simplified dynamic model of the one-on-one satellite encounter. Dynamic symmetry is assumed: both have noisy azimuth-elevation data, the criterion for each is the expected miss, and both satellites have on-off thrust inputs of controllable direction. Sample times and control times are simultaneous, and the known time of minimum range is independent of the controls. Under a set of plausible assumptions, the expected miss is minimized by the pursuer and maximized by the evader. Typical solutions are illustrated, and refinements are discussed.

Nomenclature

A_p, A_e = maximum control acceleration levels of pursuer and evader
dA, dE = error in azimuth and elevation data
dtc = control time interval
$\boldsymbol{du}_{pe}, \boldsymbol{du}_{ep}$ = error in estimate by evader and pursuer of each other's control
$\boldsymbol{dx}_p, \boldsymbol{dx}_e$ = error in estimate of vector state by pursuer and evader
$\boldsymbol{dy}_p, \boldsymbol{dy}_e$ = difference of data and predicted data of pursuer and evader
e = normalized change in miss due to evader's thrust input
F = discrete transition matrix (4×4)
G = discrete control matrix (4×4)
G_p, G_e = gains relating estimates and controls of pursuer and evader
H = output matrix (2×4)
k_p, k_e = noise covariance gains of pursuer and evader
K = Kalman gain matrix (4×2)
M = predata covariance matrix of pursuer or evader (4×4)
p = normalized change in miss due to pursuer's thrust input
P = postdata covariance matrix of pursuer or evader (4×4)
Q = driving noise covariance of pursuer or evader (2×2)
$\boldsymbol{r}$ = noise in data vector of pursuer or evader (2×1)
$\boldsymbol{r}_{pf}, \boldsymbol{r}_{ef}$ = estimates of miss by pursuer and evader (2×1)
R_p, R_e = data noise covariance of pursuer or evader (2×2)
t_f = final time, integer multiple of T
T = time interval between successive data and controls
$\boldsymbol{u}_p, \boldsymbol{u}_e$ = control accelerations of pursuer and evader
$\boldsymbol{u}_{ep}, \boldsymbol{u}_{pe}$ = estimate by pursuer of u_e and by evader of u_p
x = azimuth miss component
$\boldsymbol{x}$ = state vector (error and error rates)
x_e, y_e = evader's estimate of azimuth and elevation miss components
x_p, y_p = pursuer's estimate of azimuth and elevation miss components
y = elevation miss component
$\boldsymbol{y}_p, \boldsymbol{y}_e$ = vector data of pursuer and evader
z = range, decreasing linearly with time
θ_p, θ_e = $\tan^{-1}(r_{pf}/\sigma_{pf}) \tan^{-1}(r_{ef}/\sigma_{ef})$
σ_p, σ_e = rms error in position data of pursuer and evader

Introduction

PURSUIT-evasion differential games[1] are more complex when the data of pursuer and evader include additive noise. In this case, the pursuer P and the evader E, respectively, want to minimize and maximize their *estimates* of the performance criterion, which is the miss distance. The players in this noisy differential game are satellites, which follow straight nominal paths until minimum range. The miss is small, relative to the initial range. The initial expected miss is zero, in both azimuth and elevation directions. The thrust controls of both spacecraft are small and cannot be throttled; i.e., each thrust is either "on" or "off" at each control update time, and, when it is on, its direction must be specified.

The nominal geometry of the intercept is shown in Fig. 1, in both real space and relative space. The latter geometry shows the evader's trajectory in axes fixed to the pursuer. This shows that the miss, whether large or small, can be modified by both P and E. The two acceleration magnitudes are known to both, as is the time of minimum range. This time is independent of the control history, because only the small velocities normal to the initial range are changed by the two thrusters. The guidance algorithm to be analyzed answers the successive questions: 1) Is the estimated miss at this time accurate enough to justify an input control? and 2) Assuming the opposition has the same estimated miss, should the control thrust be zero, toward, or away from the other?

Earlier stochastic differential games[2-4] have dealt with more general characteristics of the theory and have not been illustrated by numerically explicit examples. This study is directed at finding guidance tactics for a physically motivated problem, and certain assumptions are made in order to develop these tactics. Other assumptions would lead to different results. The controls of both players will depend on the estimates of the miss vector and their uncertainties. This dependence is the novelty of the study.

Analysis

The differential-game version of the linear deterministic terminal miss problem with full information is a two-point boundary-value problem. Its solution requires that P and E apply thrust controls parallel to each other and to the final range vector, the magnitude of which is the quantity being optimized. If the initial condition can be brought to a collision course, and the controls are linear functions of the projected

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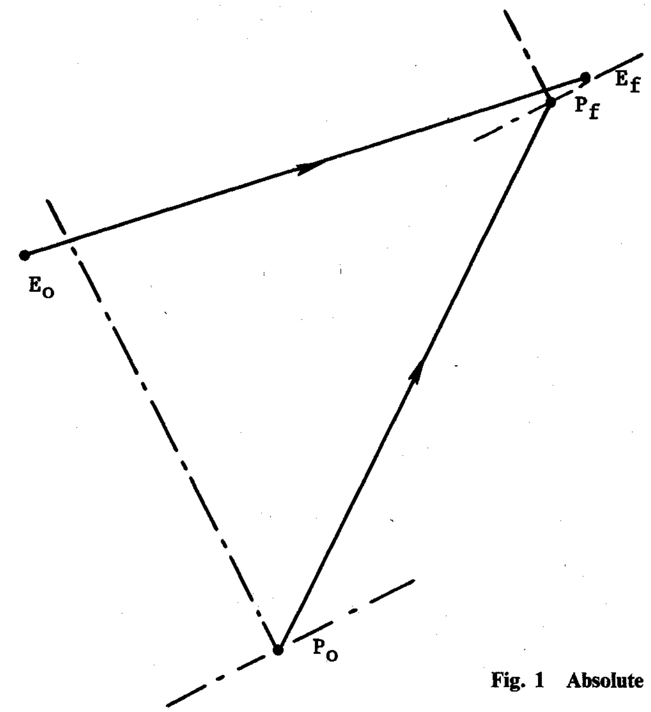

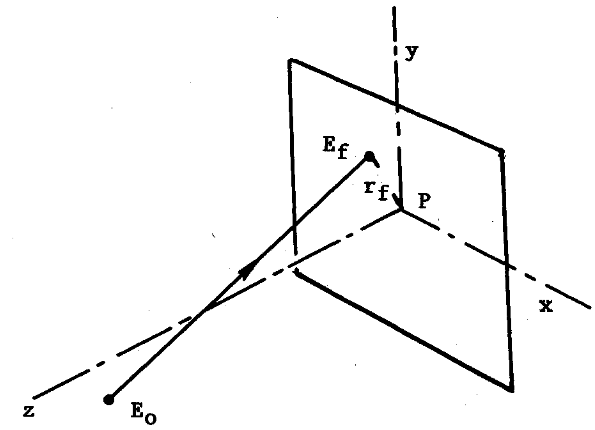

Fig. 1 Absolute and relative motion.

miss, P's larger control can guarantee an intercept when control dynamics are ignored, because any input by E is immediately canceled by P. Otherwise, the optimal *min–max* miss path will occur in the plane defined by the initial relative position and velocity vectors. The optimal trajectory then has the interpretation that, if P (or E) were at any time to use a nonoptimal control, the miss would increase (or decrease) to favor E (or P).

The stochastic version of the same problem is more complex, since now the game is not zero sum, and instead P and E must optimize their *estimates* of the miss. This means that "wrong" controls may, at any time, be applied by either, to the advantage of the other. In the discrete system version studied here, the controls of both players are updated simultaneously as functions of the current estimated relative position and velocity, the uncertainties in these estimates, the maximum control amplitudes, and other parameters.

As noted by Isaacs,[1] the stochastic differential game must include the possibility of random evasive maneuvers. In his words, "... as E's only objective is to degrade P's information, it is clear that randomized strategies are essential." Note the analogy with evasive tactics in such games as soccer and basketball. Because the two controls are specified simultaneously over the next time interval, such random tactics make a zero miss unattainable, whatever the accuracy of P's data. Instead, the expected miss is the performance criterion of each, and, in this problem, three generalizations of the nominal zero-miss problem can be cited. First, when the time to go is large, the uncertainty in the miss may be much larger than the miss estimate itself. In this case, deterministic control for P (e.g., an acceleration vector of fixed duration) is not appropriate despite its large influence on the miss at this time. Second, when the time to go is small, the uncertainties are also small, but now the maximum controls have a small effect on the miss, since the displacement due to the control is proportional to the square of the time to go. Third, the noise in the data implies that the game is not zero sum, and that P and E can at best optimize their *estimates* of the miss. Each assumes that, at any time, the other has the same estimate and covariance of the miss.

The simplest model of the two-satellite encounter involves a remarkable number of independent parameters. Those of most apparent importance are listed in Table 1. All of these parameters can influence the miss, which makes general conclusions for this nonlinear problem essentially impossible.

Table 1 Number of independent parameters

Initial position and velocity vectors normal to the initial range (r,v)	4
Initial estimates of these vectors by P and E (r_p,v_p), (r_e,v_e)	8
Initial covariances of estimates (M_p,M_e)	8
Mean-square noise in data (R_p,R_e)	2
Time-to-go, control time interval, and data interval (t_f,dtc,T)	3
Maximum control levels (A_p,A_e)	2
Mean-square uncertainties by P and E of the other's control (Q_p,Q_e)	2
Total	29

The filter and control algorithms for P and E run in parallel and are modeled by treating as data the current relative position, plus noise of zero mean value. The state is of four components, which are the position and velocity of E relative to P, normal to the initial line of sight, as shown in Fig. 1. The nominal relative motion is a collision course, with E moving toward P, down the z axis. Gravity is ignored by assuming that P and E are so near that its effects on both are equal. The discrete dynamic equations are written for the azimuth and elevation position and velocity vectors (Fig. 1) describing E's motion relative to the z axis over a sample interval, T:

$$R(n+1) = R(n) + V(n)T + A(n)(T - dtc/2)dtc$$

and

$$V(n+1) = V(n) + A(n)dtc \tag{1}$$

or, in terms of the four-component state, $x = [x, \dot{x}, y, \dot{y}]$,

$$x(n+1) = Fx(n) + G[u_e(n) - u_p(n)] \tag{2}$$

where

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$G = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \\ 0 & T^2/2 \\ 0 & T \end{bmatrix}$$

In Eq. (1), A is the difference between the evader's input acceleration and the pursuer's input acceleration over the control interval dtc. The actual dynamics do not include a random forcing "noise," but since neither P nor E knows the control being used by the other, their covariance equations must include a term representing this ignorance. Their state equations also include the estimate by each of the other's control at this time. The homogeneous azimuth and elevation equations are uncoupled.

In vector form, P's a priori estimate of the updated state has the time update

$$x_p(n+1|n) = Fx_p(n|n) + G[u_{ep}(n) - u_p(n)] \tag{3}$$

A similar equation describes the time variation of E's estimate, which includes an estimate of P's control.

The error in the pursuer's estimate of the state is $dx_p = x - x_p$, which varies with time as

$$dx_p(n+1|n) = Fdx_p(n|n) + Gdu_e(n) \tag{4}$$

since u_p does not affect the error in the estimate. This equation is used to develop the time update of P's covariance matrix.

The data update of P's vector estimate has the Kalman filter format,

$$x_p(n|n) = x_p(n|n-1) + K(n)[y(n) - y_p(n|n-1)] \tag{5}$$

The vector

$$y_p(n|n-1) = Hx_p(n|n-1)$$

is P's predicted data at time n, based on data to time $n-1$, and $K(n)$ minimizes the covariance of the error in this estimate.

The difference between the azimuth-elevation data and its predicted value, shown in the brackets in Eq. (5), is

$$dy_p(n) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ 0 & 0 & 1/z & 0 \end{pmatrix} dx_p(n) + \begin{pmatrix} dA(n) \\ dE(n) \end{pmatrix} \tag{6}$$

or

$$dy_p(n) = Hdx_p(n) + r(n) \tag{7}$$

The H matrix depends on the z component of the range, assumed known to P and E.

The uncertainty in the estimated control of each appears in the noise covariance over the next time interval, which depends on the control capability of the other. That is, the time update of the previous covariance, as implied by Eq. (4), is

$$M(n+1) = FP(n)F' + GQ(n)G' \tag{8}$$

The data error covariance $Q(n)$ is computed as described below, and the post-data covariance is

$$P(n) = [I - K(n)H]M(n) \tag{9}$$

where the Kalman estimation gain matrix is

$$K(n) = M(n)H'[HM(n)H' + R(n)]^{-1} \tag{10}$$

The noise in the data of Eq. (5) has the covariance $R(n) = E[r(n)r(n)']$, which, in this case, is a 2×2 matrix having the mean-square uncertainties in azimuth and elevation data angles on the diagonals, both presumed known.

Control Algorithm

The controls used by P and E depend on the uncontrolled miss estimates, as extrapolated from the current state estimate and on their covariances. In qualitative terms, neither considers a miss estimate good enough to be taken seriously unless the magnitude of the miss is "somewhat greater" than its uncertainty. For P, a deterministic control is computed when the projected uncontrolled miss is more than G_p times the projected final rms position error. This gain is chosen by P, typically between 1 and 3, and it defines a "good" estimate of the miss in terms of the simultaneous rms $(1-\sigma)$ uncertainty. The evader similarly defines a good estimate in terms of the gain G_e, such that E's deterministic control is calculated only if

$$r_{ef} \geq G_e \sigma_{ef} \tag{11}$$

Mixing of the deterministic and random components of E's control is discussed below.

Both P and E estimate the other's control, and the uncertainty in this estimate acts as a noise covariance over the next time interval. P takes as the current covariance of the process noise the quantity $Q_p = (k_p A_e)^2$, where A_e is presumed known to P. A similar expression gives the process noise covariance in E's equations.

When the estimate and its rms uncertainty are above the line indicated in Fig. 2a, P assumes that E has a large and accurate estimate of the miss, and that the size of E's random control is relatively low. The gain k_p is equal to 1 when the estimate is below the control gain line, because, in this case, P's estimate (and, by P's assumption, E's estimate) of the miss is both small and inaccurate. Therefore, the process noise gain for P varies between 0 and 1, according to

$$k_p = \begin{cases} \dfrac{\pi/2 - \theta}{\pi/2 - \theta_p} & \text{if} \quad r_{pf}/\sigma_{pf} > G_p \\ 1 & \text{if} \quad r_{pf}/\sigma_{pf} < G_p \end{cases} \tag{12}$$

Similar considerations determine E's gain k_e for P's control covariance, but, in this case, E is least "sure" of P's control when the estimate and its uncertainty are below the gain line of slope G_p, because the control region is bounded by this line. The noise covariance is thus assumed to vary bilinearly, as shown in Fig. 2b.

This second curve also requires comment. A very large or very accurate estimate of the uncontrolled miss implies that the effective E's uncertainty in P's control is small, and, therefore, E's state "process noise" is small, and E assumes that P will use the control appropriate to the current geometry. At the other extreme, a small or inaccurate estimate by E of the miss implies that E is less sure of P's control, so the noise is given a larger covariance.

E's miss estimate r_{ef} defines a control direction, which is modified by a random component of half-amplitude $\pi\sigma_{ef}/$

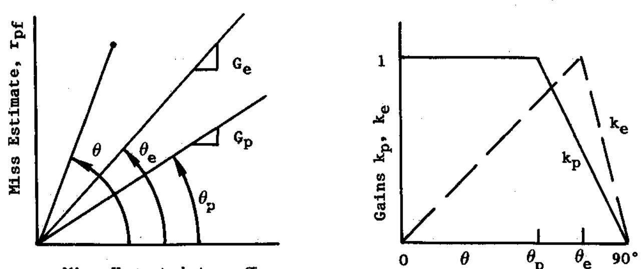

Fig. 2 Noise gain computation: a) estimates and uncertainties; b) noise covariance gains.

$(\sigma_{ef} + r_{ef})$. That is, if the estimate is inaccurate, then $\sigma_{ef} \gg r_{ef}$, and most of E's control is random. However, the random component approaches zero with the ratio of uncertainty to magnitude of the estimated miss, implying that E plays deterministically if the miss estimate is very large or very accurate. Evasive tactics in pursuit-evasion sports provide some justification for this logic.

When either estimate is accurate enough to justify a deterministic control, its direction is found as follows. The elements of a 3 × 3 projected miss matrix are determined. These nine misses correspond to controls by either player of nominal velocity change toward or away from the other, and no control. The change in miss due the control A_p over the next control interval dtc, when the time to go is t_f, is found to be

$$p(t_f, dtc) = A_p(t_f - dtc/2)dtc \tag{13}$$

A similar expression gives E's contribution e to the miss; it is assumed that $A_e < A_p$ so that P can control the direction of the relative motion at any update time.

The controls of the players are specified at each time interval after each has updated his estimated final relative position. The bang-bang control makes the algorithm for each dependent on the miss-matrix elements, which has the normalized form shown in Table 2.

The unit at the center of this array corresponds to the uncontrolled miss, so that the elements have been normalized by this distance. Since the data are at least slightly different for P and E, so are the estimates p and e. The miss matrices for P and E therefore always differ, although their expected values are the same at any time, since the noise is unbiased. The importance of the miss matrix is that P would like to minimize, and E to maximize the miss resulting from the controls of both. Assuming that strategies are picked simultaneously, P's control is implied by the column with the smallest maximum miss. The evader chooses the control corresponding to the row with the largest minimum miss.

Except for contrived problems having identical misses for more than one control pair, the "optimal" min-max pair of controls is given by comparing three parameters for both players. This simple logic can produce unexpected results, however. For example, if $p = 2.2$ (a control pulse along the predicted miss direction changes the miss by this factor relative to the nominal) and $e = 0.8$, P's control effectiveness is almost triple that of E's, and appears as Table 3.

The second column has the smallest maximum, so P's control is 0. This is because the uncontrolled miss is so small that any other control of the pursuer causes a large overshoot. Similarly, the second row has the largest minimum, so E should also coast until the next data sample. Here we are assuming identical matrices for both; actually, they would be slightly different due to data noise. However, if E knows that P has chosen 0, E does better by choosing the third row (1.8). If P then uses this knowledge, his tactic changes to column 1 (0.4), following which E switches to row 1 (2.0), and P to column 2 (0.7). The resulting limit cycle of indefinite controls is discussed in Ref. 5. In the numerical results to follow, both controls would be zero, because no assumptions are made by either as to deviations by the other from this elementary strategy.

Table 2 Bang-bang control matrix

		Control P		
		$-p$	0	$+p$
	$-e$	$\lvert 1-p-e \rvert$	$\lvert 1-e \rvert$	$1+p-e$
Control E	0	$\lvert 1-p \rvert$	1	$1+p$
	$+e$	$\lvert 1-p+e \rvert$	$1+e$	$1+p+e$

Table 3 Control matrix, $p = 2.2$, $e = 0.8$

		Control P		
		$-p$	0	$+p$
	$-e$	2.0	0.2	2.4
Control E	0	1.2	1.0	3.2
	$+e$	0.4	1.8	4.0

Numerical Results

Because of the large number of parameters in this nonlinear problem, only a few results are shown here. The problem solution is illustrated by the transients in azimuth and elevation and the simultaneous histories of position rms uncertainties of both P and E. The more interesting cases are those for which the data are very poor, but now the paths depend strongly on the specific noise samples in the data of each. Monte Carlo averaging of such simulated encounters can smooth the results for comparative purposes.

A nominal case is defined in which E's data noise is 2.5 times larger (rms) than P's. The nominal trajectory begins with a moderate range rate normal to the line of sight, and initial estimates of P and E lead to the "nearby" estimated miss shown in Fig. 3. The pursuer has a larger control (3 vs 1 m/s^2) and the 100-s trajectory can be modified by either at 5-s intervals. The uncontrolled trajectory has a final miss of 3640 m. Relative position data for P and E are in error by 2 m and 5 m (rms), respectively. The transient rms miss uncertainties of pursuer and evader are shown in Fig. 4. The subsequent path is shown in Fig. 5 for several noise sequences. The paths are remarkably different because the nominal miss is small and the data noise is both large and uncorrelated.

A typical sensitivity plot is shown in Fig. 6. For the given initial condition, gains G_p and G_e are assigned values of 1, 2, and 3. The resulting mean miss is shown as the dependent

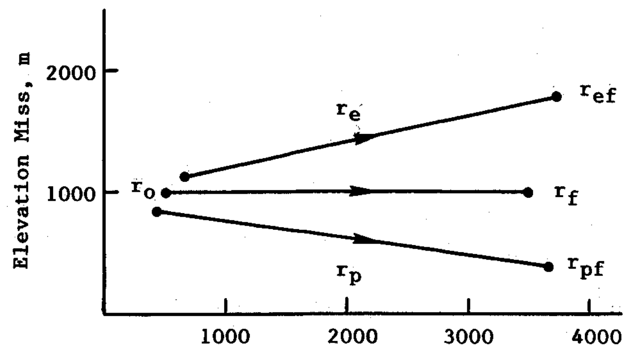

Fig. 3 Nominal trajectory and initial estimates.

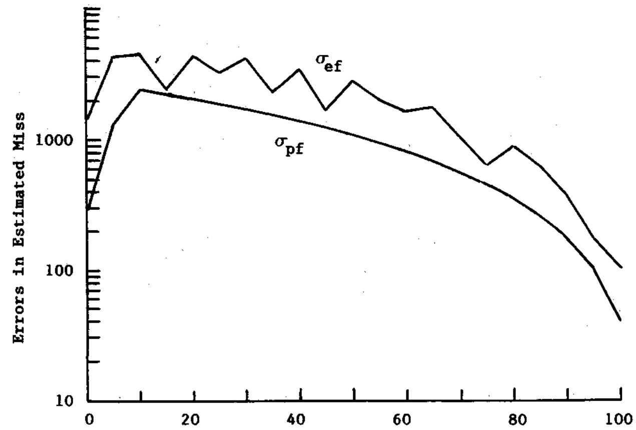

Fig. 4 Root-mean-square uncertainties for pursuer and evader.

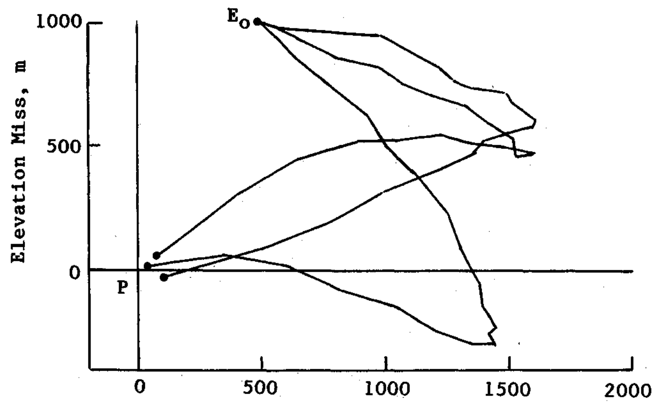

Fig. 5 Trajectory variations with data noise.

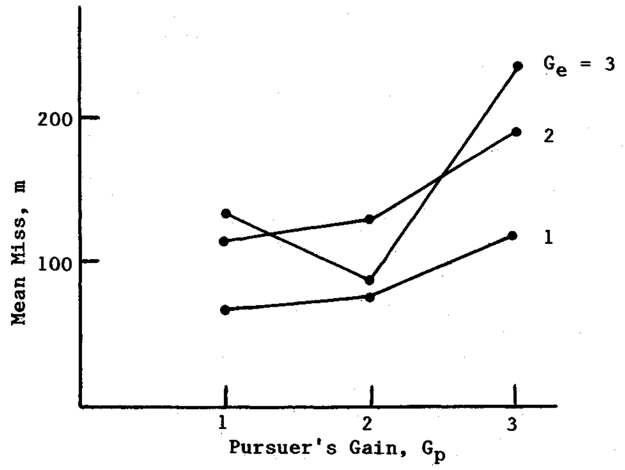

Fig. 6 Miss variations with gains G_p and G_e.

variable. The two bang-bang controls, or "the mean value of the resulting miss," and the data noise result in a ragged dependence of miss on the parameters, but the "min-max" gains of both P and E seem to be approximately equal to 2.

The variation of miss with the pursuer's control level is shown in Fig. 7 for $G_p = G_e = 2$. The points in both Fig. 6 and Fig. 7 are found as the average miss from 20 trajectories differing only by the noise samples added to the data. Similar variations can be found with respect to any of the other parameters tabulated earlier in the Analysis section.

Conclusions

A "solution" has been found for this stochastic differential game by determining the saddlepoint min-max element of a 3×3 miss matrix, to which a unique pair of controls corresponds. As is well known, stochastic differential games have features that do not allow closed-loop solutions to be determined, even when the dynamics are linear. In the present case, the on-off character of the controls allows control computation from a small number of possibilities. The principal features of the algorithm are that the controls for each player account for the uncertainty in the current estimates and for the simultaneous control of the other. The miss estimate and its uncertainty for each player at each update time imply the deterministic controls. The evader's control is partly random, which varies in magnitude with the ratio of estimated miss and uncertainty in this estimate. The algorithms for both are min-max in an open-loop sense, but the finite control combinations assumed do not account for random control contributions.

No comparable implementation of min-max algorithms has been found in the literature. The form of control and the

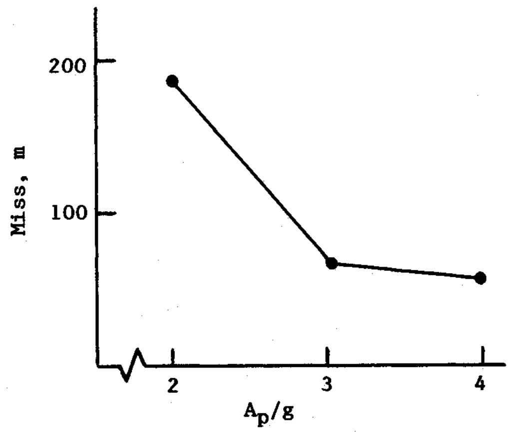

Fig. 7 Miss variation with control level A_p.

control gains are specified rather than derived, so optimality cannot be claimed for the general initial condition. Other reasons for nonoptimality are conjectured to be the following: 1) the pursuer's control does not include a random component to mislead the evader; 2) the miss is a bilinear function of the controls, thus ignoring the possibility of multiple maneuvers out of the plane of the miss vector; and 3) the optimal control for each should be specified from a statistical distribution.[5] Thus, for a specific set of parameters, E might require as control $+e$ 60% of the time, $-e$ 30% of the time, and zero 10% of the time. The proportions would depend on the covariances in the estimate of the miss vector and on the covariances of E's estimates of P's control at that time. This level of complexity is not appropriate to the current rudimentary understanding of this type of problem.

The separation principle[6] is relevant to the optimization of linear Gaussian stochastic systems with quadratic performance index. The present application is only partly analogous, because both players model the uncertainty in the other's control as normally distributed noise, which is not the case. For nonlinear stochastic dynamics,[3] the optimal control system does not generally separate into an optimal filter in series with an optimal control, as assumed here. On the other hand, local linearization as used in this implementation gives numerically reasonable results for the parameter values assumed.

A practical generalization of the model assumed here would replace the bang-bang thrusters by a bounded linear region, perhaps including a dead-zone. The penalty for this generalization is an increase in the already large number of descriptive system parameters. The objective has been to develop and test a plausible and unbiased pair of control laws for a simplified version of an actual noisy differential game. Limited results suggest that practical performance sensitivities can be found by the methods described.

References

[1]Isaacs, R., *Differential Games—A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*, Wiley, New York, 1965, Chap. 12.

[2]Behn, R. D. and Ho, Y-C., "On a Class of Linear Stochastic Differential Games," *IEEE Transactions on Automatic Control*, AC-13, June 1968, pp. 227–239.

[3]Ho, Y-C., "Optimal Maneuver and Evasion Strategy," *Journal of SIAM Control*, Vol. 4, May–June 1966, pp. 421–428.

[4]Rhodes, I. B. and D. G. Luenberger, "Differential Games with Constrained State Information," IEEE *Transactions on Automatic Control*, AC-14, Feb. 1969, pp. 29–39.

[5]Bryson, A. E., Jr. and Ho, Y-C., *Applied Optimal Control—Optimization, Estimation and Control*, Blaisdell, Waltham, MA, 1969, Chap. 9.

[6]Gelb, A. (ed.), *Applied Optimal Estimation*, M.I.T. Press, Cambridge, MA., 1974, p. 365.